NOTE ON THE THEORY OF FUNCTIONS OF A REAL VARIABLE.*

By W. H. Echols, Charlottesville, Va.

§ 1. The application of the Differential Calculus to investigations in the Theory of Functions of a real variable depends on the formation and the character of the nth derivative. This marks the limit of its applicability. The following note is intended to be illustrative, at the same time, of the process and of its limitations.

I.

Foundation.

§ 2. Let fx represent an explicit function of the variable x. The function fx is said to be *finite* for $x = a$, when fa is not infinite. fx is said to be *uniform* for $x = a$, when fa has a single determinate value.

The uniform and finite function fx is *continuous* at a when it is possible to find a finite number h, such that

$$f(a + \theta h) - fa \qquad\qquad -1 < \theta < +1$$

is less than an arbitrarily small assigned number δ, and

$$\mathop{\pounds}_{h=0} f(a + \theta h) = fa\,.$$

The function is progressively continuous at a when $0 < \theta < +1$, and regressively continuous at a when $-1 < \theta < 0$.

The function fx is said to be uniform, finite, and continuous throughout the interval from $x = a$ to $x = \beta$, when ($\beta > a$) it is uniform, finite, and continuous for every value of x that is equal to or greater than a, and equal to or less than β.

If fx be u. f. c.† throughout the finite interval $(a\beta)$, then it follows from the definitions, that

$$f(a + h) = fa + \sigma\,,$$

where a and $a + h$ are values in the interval, and as h converges uniformly to zero, σ converges continuously to zero.

* This note is intended to be a somewhat critical examination of the fundamental principles which underlie the general method for expansion of real functions in series, given in a crude note written January, 1892, "On Certain Determinant Forms and their Applications."

† Uniform, finite, and continuous.

§ 3. Complete difference of a sequence :—

Let there be a sequence of the $n+1$ terms

$$A_0, \quad A_1, \quad \ldots, \quad A_n.$$

Form a new sequence of $n+1$ terms, thus :

$$A_0, \quad A_1 - A_0, \quad \ldots, \quad A_n - A_{n-1}.$$

From this form a new sequence, beginning with the second term and subtracting each term from the one which follows it. Continue this operation until there have been formed n new sequences from the first one. The last sequence of these is called the *complete difference* of the first sequence. Its terms are

$$A_r - C_{r,1} A_{r-1} + \ldots + (-1)^r C_{r,r} A_0. \qquad (r = 0, \ldots, n)$$

§ 4. We assume a one-to-one correspondence between the variable x and and the x-axis of Cartesian coordinates, and that the function is represented by the ordinate. We speak of the point a as the point on the x-axis for which $x = a$, and of the point fa as the point established by the coordinates (a, fa).

Consider the sequence whose terms are

$$f(a + rh). \qquad (r = 0, \ldots, n)$$

The points $a + rh$ being in the u. f. c. interval of fx.

The complete-difference of this sequence is the sequence

$$f(a + rh) - C_{r,1} f(a + \overline{r-1}h) + \ldots + (-1)^r fa. \quad (r = 0, \ldots, n)$$

This expression we call the rth *difference* of the function fx at the point a. It is *progressive* or *regressive* according as h is positive or negative. We symbolize the nth general difference of fx at a, by $\Delta^{n|h} fa$.

We have,

$$\begin{aligned} \Delta^{n|h} fa &= f(a + nh) - C_{n,1} f(a + \overline{n-1}h) + \ldots + (-1)^n fa \\ &= fa \left[\frac{f(a+nh)}{fa} - C_{n,1} \frac{f(a + \overline{n-1}h)}{fa} + \ldots + (-1)^n \right] \\ &= fa\,[(1 + \sigma_n) - C_{n,1}(1 + \sigma_{n-1}) + \ldots + (-1)^n] \\ &= \sigma^n fa. \end{aligned}$$

For

$$f(a + rh) = fa + \delta_r,$$

where δ_r has a value which is equal to zero when $h = 0$, and

$$\frac{f(a+rh)}{fa} = 1 + \frac{\delta_r}{fa} = 1 + \sigma_r,$$

where $\sigma_r = 0$ when $h = 0$. Also, there must exist some value σ such that $\sigma = 0$ when $\delta_r = 0$, $\sigma_r = 0$, $h = 0$ and is determined by

$$[(1 + \sigma) - 1]^n = (1 + \sigma_n) - C_{n,1}(1 + \sigma_{n-1}) + \ldots + (-1)^n .$$

From these results we infer that the nth difference of the continuous function at a becomes a vanishing value of nullitude n as h converges to zero.

§ 5. The ratio

$$\frac{\Delta^{n|h}fa}{h^n},$$

we call the nth *difference-ratio* of the function fx at a, progressive or regressive according as h is positive or negative, and having a different value in each case. This ratio, as we have seen, has the form

$$\frac{\Delta^{n|h}fa}{h^n} = \left[\frac{\sigma}{h}\right]^n fa ,$$

wherein σ becomes infinitesimal at the same time with h.

It is through this form that we propose to separate u. f. c. functions into classes.

The ratio σ/h, as h converges to zero, may become indeterminate; functions which yield this result we reject for the present. We also set aside that class of continuous functions which are such that σ becomes infinitesimal of lower order than h, which makes the limit of the ratio σ/h infinite. We consider u. f. c. functions to be *monogenic* functions, when the limit of the ratio σ/h is determinate, uniform, and not infinite (n finite) as h converges to zero, whatever be the sign of h.

The u. f. c. function fx is monogenic at a when (for $\pm h$)

$$\mathop{\pounds}_{h=0} \frac{\Delta^{n|h}fa}{h^n}$$

is uniform and finite for a finite value of n.

A function which is uniform, finite, continuous, and monogenic at a is said to be *holomorphic* at a. The function is holomorphic throughout the interval $(a\beta)$ when it is holomorphic at each point in the interval.

§ 6. We write the limit of the nth difference-ratio

$$\mathop{\pounds}_{h=0} \frac{\Delta^{n|h}fa}{h^n} = f^n a ,$$

and call it the nth *derivative-ratio*, or simply the nth *derivative* of the function at a.

The difference-ratios higher than the first, while peculiarly suited to the above suggested classification are not well suited for the calculation of the derivatives. The first derivative is determined as the limit of the first difference-ratio, thus :

$$f'a = \mathop{\pounds}_{h=0} \frac{f(a+h) - fa}{h},$$

and the successive derivatives from the form

$$f^{n+1}a = \mathop{\pounds}_{h=0} \frac{f^n(a+h) - f^n a}{h}.$$

For

$$f^n a = \mathop{\pounds}_{h=0} \frac{\Delta^{n|h} fa}{h^n},$$

$$f^n(a+h) = \mathop{\pounds}_{h=0} \frac{\Delta^{n|h} f(a+h)}{h^n}.$$

Therefore

$$\frac{f^n(a+h) - f^n a}{h} = \mathop{\pounds}_{h=0} \frac{1}{h^{n+1}} [\Delta^{n|h} f(a+h) - \Delta^{n|h} fa]$$

$$= \mathop{\pounds}_{h=0} \frac{\Delta^{n+1|h} fa}{h^n},$$

since

$$C_{n,r} + C_{n,r-1} = C_{n+1,r}.$$

Passing to the limit, we have

$$\mathop{\pounds}_{h=0} \frac{f^n(a+h) - f^n a}{h} = f^{n+1}a.$$

II.

Fundamental Theorems. Holomorphic Functions.

§ 7. Theorem I. *If fx be a holomorphic function throughout the interval $(\alpha\beta)$, its first derivative is a holomorphic function throughout the interval $(\alpha\beta)$.*

$f'x$ is uniform and finite at a, since by definition

$$\frac{f(a+h) - fa}{h}$$

converges ($h = 0$) to a single determinate finite value $f'a$. $f'x$ is continuous at a, because

$$f'(a+h) - f'a$$

must become infinitesimal when h becomes infinitesimal, since

$$\frac{f'(a+h) - f'a}{h}$$

converges ($h = 0$) to the uniform finite limit $f''a$. Moreover, the successive derivatives of $f'x$ are the derivatives of fx; consequently, $f'x$ is monogenic at a. Being uniform, finite, continuous, and monogenic at any point a in $(\alpha\beta)$, $f'x$ is holomorphic throughout the interval.

COROLLARY. A function which is holomorphic throughout a finite interval, has an unlimited number of successive derivatives each of which is holomorphic throughout the interval.

§ 8. A continuous variable x is said to vary *uniformly* from α to β, when as x passes from the value α to the value β, it takes during the passage any value that is equal to or greater than α and equal to or less than β, and but *once*.

A continuous function fx is said to vary uniformly from $f\alpha$ to $f\beta$, or through the interval $(\alpha\beta)$, when as x varies uniformly through $(\alpha\beta)$ the function takes during the passage any value between the values $f\alpha$ and $f\beta$ (these included) but *once*.

It follows therefore that a continuous function which varies uniformly through an interval, has algebraically an increasing or decreasing value throughout the interval according as

$$f(x+h^2) - fx$$

is positive or negative, however small we take h. Or, as the derivative

$$f'x = \mathop{\pounds}_{h=0} \frac{f(x+h^2) - fx}{h^2},$$

is positive or negative throughout the interval.

THEOREM II. *In any arbitrarily small finite interval in the interval $(\alpha\beta)$ of a holomorphic function fx, the function must vary uniformly.*

For

$$f'a = \mathop{\pounds}_{h=0} \frac{f(a+h) - fa}{h}$$

must be a uniform, determinate, finite limit, and $f'x$ is continuous throughout

$(\alpha\beta)$. Hence

$$\frac{f(a+h)-fa}{h}=\frac{f(a+\theta h)-fa}{\theta h} \qquad 0<\theta<1$$

must be arbitrarily small with h. This cannot be so (unless $f'a=0$) when for any θ we have

$$f(a+\theta h)=fa\,.$$

Therefore fx must be an increasing or decreasing function from a to $a+h$.

The theorem (Cauchy's) which justifies these definitions is as follows :

THEOREM III. *If V be any fixed value whatever between fa and $f\beta$, then the continuous function fx must take the value V for some value u, of x, between a and β.*

Suppose $fa<f\beta$ (algebraically). Let $\beta>\alpha$, and divide the interval $(\alpha\beta)$ into 10 equal parts, each equal to h. Consider the sequence

$$fa,\quad f(a+h)\,,\quad \ldots,\quad f(a+nh)=f\beta\,.$$

If any one of these values be identical with V, the theorem is proved. Otherwise, let fa_1 be the last term of the sequence, proceeding from a, which is less than V. Let $f\beta_1$ be the last term of the sequence, proceeding from β, which is greater than V. Divide the new interval $(\alpha_1\beta_1)$ into 10 equal parts h_1. Suppose the function does not take the value V at any of these points of division. Then, let in like manner $fa_2<V$ and $f\beta_2>V$, define a new interval $(\alpha_2\beta_2)$ with which we proceed as before. Continue this operation until either fx takes the value V at one of the subdivision points, or we reach an interval $(\alpha_n\beta_n)$, such that

$$f\alpha_n<V<f\beta_n\,.$$

Now

$$\beta_n-\alpha_n=10\,h_n\,,$$

and

$$\beta-\alpha=10\,h=10^2h_1=\ldots=10^{n+1}h_n\,.$$

Therefore

$$h_n=\frac{\beta-\alpha}{10^{n+1}}\,,$$

and

$$\beta_n-\alpha_n=\frac{\beta-\alpha}{10^n}\,.$$

α_n continually increases but never reaches β ; therefore it has a limit. β_n continually decreases but never reaches α, it therefore has a limit. The difference $\beta_n-\alpha_n$ can be made as small as we choose by sufficiently increasing n ;

consequently they converge to the same limit u, which is $\alpha < u < \beta$. But the value V always lies between $f\alpha_n$ and $f\beta_n$, and since the function fx is continuous we can make the difference between $f\alpha_n$ and $f\beta_n$ as small as we choose by making $\beta_n - \alpha_n$ sufficiently small. Consequently V is the limit to which converge $f\alpha_n$ and $f\beta_n$ as α_n and β_n converge to the limit u. Therefore, we have

$$fu = V. \qquad \alpha < u < \beta$$

COROLLARY. If a continuous function has opposite signs at the ends of an interval $(\alpha\beta)$, then must there be some value u such that

$$fu = 0. \qquad \alpha < u < \beta$$

THEOREM IV. *If a holomorphic function fx has equal values at two points a and b in its interval of holomorphism, then its derivative must have a zero between a and b.*

If fx is constant for any finite portion of the interval (ab), the theorem is proved. Otherwise, fx must be either an increasing or decreasing function as we proceed from a to b. Now fx cannot continue to be an increasing or decreasing function throughout the interval from a to b, for, if so, it would be impossible for fb to equal fa. Therefore at some point x_1 in the interval, fx must be an increasing function, and at some point x_2 a decreasing function. Since the function is holomorphic, the derivative $f'x$ must have opposite signs at the points x_1 and x_2. Since the derivative is a continuous function between x_1 and x_2, it must have a zero in the interval (x_1x_2). Consequently, we have

$$f'u = 0. \qquad a < u < b$$

COROLLARY. If a and b are zeros of the function, its derivative must have a zero between a and b.

THEOREM V. *If a holomorphic function has $n + 1$ zeros in its interval $(\alpha\beta)$, then the nth derivative must have a zero between the greatest and least of the zeros of the function.*

Let the zeros of fx be $a_0, a_1, \ldots, a_n$ taken in increasing order. The first derivative $f'x$ has a zero in each of the n intervals $(a_0a_1), \ldots, (a_{n-1}\, a_n)$, say $b_0, \ldots, b_{n-1}$. But $f'x$ is a holomorphic function in $(\alpha\beta)$, therefore its derivative $f''x$ must have a zero in each one of the $n - 1$ intervals $(b_0b_1), \ldots, (b_{n-2}\, b_{n-1})$, say $c_0, \ldots, c_{n-2}$. In like manner $f'''x$ has a zero in each of the $n - 2$ intervals $(c_0c_1), \ldots, (c_{n-3}\, c_{n-2})$. In this way we continue until f^nx must have a zero, u, in the one remaining interval, or

$$f^nu = 0. \qquad a_0 < u < a_n$$

COROLLARY. The same result is true if fx has any $n+1$ equal values in the interval.

§ 9. LEMMA. Let $$a_r = a + rh\,, \qquad (r = 0, \ldots, n)$$
then
$$(-1)^n \Delta^{n|h} fa_0 = \frac{\begin{vmatrix} fa_0 & 1 & a_0 & \ldots & a_0^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n & 1 & a_n & \ldots & a_n^{n-1} \end{vmatrix}}{\zeta^{\frac{1}{2}}(a_1, \ldots, a_n)}\,.$$

For, expanding the second member by the first column of the numerator it becomes

$$\sum_0^n{}_r (-1)^r fa_r = \sum_0^n{}_r (-1)^r \frac{\zeta^{\frac{1}{2}}(a_0, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n)}{\zeta^{\frac{1}{2}}(a_1, \ldots, a_n)} fa_r$$

$$= \sum_0^n{}_r \frac{(a_0 - a_1)\ldots(a_0 - a_{r-1})(a_0 - a_{r+1})\ldots(a_0 - a_n)}{(a_r - a_1)\ldots(a_r - a_{r-1})(a_r - a_{r+1})\ldots(a_r - a_n)} fa_r$$

$$= \sum_0^n{}_r (-1)^r C_{n,r} fa_r\,.$$

But
$$\Delta^{n|h} fa_0 = (-1)^n \sum_0^n{}_r (-1)^r C_{n,r} fa_r\,,$$

which establishes the lemma.

THEOREM VI. *If a function fx is holomorphic in an interval $(\alpha\beta)$ containing the points*
$$a\,, \quad a+h\,, \quad \ldots, \quad a+nh\,,$$
then
$$\Delta^{n|h} fa = h^n f^n u\,,$$

where u lies between a and $a+nh$.

For, the holomorphic function

$$\frac{\begin{vmatrix} fx & 1 & x & \ldots & x^n \\ fa_0 & 1 & a_0 & \ldots & a_0^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n & 1 & a_n & \ldots & a_n^n \end{vmatrix}}{\zeta^{\frac{1}{2}}(a_1, \ldots, a_n)}$$

vanishes at the $n+1$ points

$$a_r = a + rh\,. \qquad (r = 0, \ldots, n)$$

Therefore its nth derivative vanishes for some point u in the interval $(a, a + nh)$, and we have

$$\frac{\zeta^{\frac{1}{2}}(a_0, \ldots, a_n)}{\zeta^{\frac{1}{2}}(a_1, \ldots, a_n)} f^n u = n!\, \Delta^{n|h} fa .$$

$$\frac{\zeta^{\frac{1}{2}}(a_0, \ldots, a_n)}{\zeta^{\frac{1}{2}}(a_1, \ldots, a_n)} = (a_0 - a_1) \ldots (a_0 - a_n)$$

$$= n!\, h^n .$$

$$\therefore \qquad \Delta^{n|h} fa = h^n f^n u .$$

COROLLARY.

$$\mathop{\pounds}_{h=0} \frac{\Delta^{n|h} fa}{h^n} = \mathop{\pounds}_{h=0} f^n u = f^n a .$$

THEOREM VII. *If a holomorphic function fx has zeros a and b in its interval $(\alpha\beta)$, and if a is a zero of each of the first n derivatives, then will the $(n + 1)$th derivative have a zero u, between a and b.*

Since the derivatives are holomorphic functions in $(\alpha\beta)$, $f'x$ has a zero u_1 in the interval (ab). $f''u$ has a zero in the interval (au_1), and so on. Finally, $f^{n+1}x$ has a zero u, in the interval (au_n). Therefore

$$f^{n+1} u = 0 . \qquad a < u < b$$

THEOREM VIII. *If a holomorphic function fx has the zeros $a_1, \ldots, a_n$ in its interval $(\alpha\beta)$. Then*

$$fx = (x - a_1) \ldots (x - a_n) \frac{f^n u}{n!} ,$$

wherein u lies between the greatest and least of the zeros and x.

Let x_0 be any fixed point in $(\alpha\beta)$. The holomorphic function

$$\begin{vmatrix} fx & 1 & x & \ldots & x^n \\ fa_1 & 1 & a_1 & \ldots & a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n & 1 & a_n & \ldots & a_n^n \\ fx_0 & 1 & x_0 & \ldots & x_0^n \end{vmatrix}$$

has the zeros $x_0, a_1, \ldots, a_n$. Its nth derivative must have a zero between the greatest and least of these. Therefore

$$f^n u\, \zeta^{\frac{1}{2}}(x_0, a_1, \ldots, a_n) - n!\, fx_0\, \zeta^{\frac{1}{2}}(a_1, \ldots, a_n) = 0$$

or

$$fx_0 = (x_0 - a_1) \ldots (x_0 - a_n) \frac{f^n u}{n!} .$$

x_0 being any point in $(\alpha\beta)$, the theorem is established.

COROLLARY 1. If a holomorphic function fx has n equal values at $a_1, \ldots, a_n$ in its interval, then

$$fx = (x - a_1) \ldots (x - a_n) \frac{f^n u}{n!} + c ,$$

where c is constant. For, in the above theorem, suppose

$$fa_1 = fa_2 = \ldots = fa_n = fa .$$

Then

$$fx = (x - a_1) \ldots (x - a_n) \frac{f^n u}{n!} + fa .$$

COROLLARY 2. If we say that the running together of the n equal values forms a multiple point of multiplicity n, then, if fx has a multiple point of multiplicity n at a, we have

$$fx = \frac{(x-a)^n}{n!} f^n u + fa .$$

In particular if the function vanishes at a, the point a is called a zero or nullitude point of nullitude n. If a be a zero of fx of nullitude n, then

$$fx = \frac{(x-a)^n}{n!} f^n u .$$

THEOREM IX. *If fx has a zero of nullitude n in its interval then this point is a zero of nullitude $n-1$ of the derivative of fx.*

Let fx have the zeros $a_1, \ldots, a_n$. $f'x$ has a zero in the $n-1$ intervals $(a_1 a_2), \ldots, (a_{n-1} a_n)$. By the preceding theorem we have

$$f'x = (x - u_1) \ldots (x - u_{n-1}) \frac{f^n u'}{(n-1)!} .$$

As $a_1, \ldots, a_n$ converge to a, so also do $u_1, \ldots, u_{n-1}$.

Therefore in the limit

$$f'x = \frac{(x-a)^{n-1}}{(n-1)!} f^n u' ,$$

u' between x and a. We infer from this result that we may differentiate

$$fx = \frac{(x-a)^n}{n!} f^n u ,$$

as though u were constant.

THEOREM X. *A holomorphic function fx cannot have an infinite number of equal values uniformly distributed in a finite interval unless the function is constant throughout the interval.*

Let fx be holomorphic throughout (ab), and let $fa_r = fa$,

$$a_r = a + rh\,, \qquad (r = 0, \ldots, m)$$

so that $h = (b - a)/m$ vanishes when $m = \infty$.

In virtue of the holomorphic character of fx and its derivative, we must have

$$\frac{fx - fa_r}{x - a_r} - \frac{fa_{r+1} - fa_r}{h}$$

less than an arbitrarily small assigned value δ (which converges to zero when h converges to zero) when h is arbitrarily small, for all values of x between a_r and a_{r+1}, whatever be the value of $r < m$. The second term of this difference is zero. Therefore we must have

$$\operatorname{mod} \frac{fx - fa}{x - a_r} < \delta\,.$$

Wherever x be taken in (ab) we always have a value $x - a_r$, less than h, such that in absolute value,

$$fx - fa < \delta\,(x - a_r)\,, \quad \text{or} \quad fx - fa < \delta h\,.$$

Hence when h converges to zero ($m = \infty$) we have in the limit

$$fx = fa\,,$$

for all points in the interval (ab).

COROLLARY 1. It follows that all the successive derivatives of fx vanish for all values of x between a and b.

For the first derivative $f'x$ has a zero u_r in the interval $(a + rh, a + \overline{r+1}\,h)$ $(r = 0, \ldots, n)$ and vanishes an infinite number of times throughout (ab). And so for the successive derivatives.

COROLLARY 2. We may attempt to investigate the value of fx at any point x_0 in $(a\beta)$ not in (ab) when fx is constant in (ab), as follows :—

Consider the function

$$Fx \quad \begin{vmatrix} fx & 1 & x & \ldots & x^{n+1} \\ fa_0 & 1 & a_0 & \ldots & a_0^{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n & 1 & a_n & \ldots & a_n^{n+1} \\ fx_0 & 1 & x_0 & \ldots & x_0^{n+1} \end{vmatrix},$$

wherein $a_r = a + rh$ $(r = 0, \ldots, n)$ and $a_n = b$, $a_0 = a$. x_0 some point in $(\alpha\beta)$ not in (ab) arbitrarily fixed. Fx has the same interval of holomorphism as fx, since Fx is made up of fx and a rational integer of degree $n + 1$. Let

$$fa = fa_1 = fa_2 = \ldots = fa_n .$$

The function Fx has the value zero at the $n + 1$ points a_r. Therefore its nth derivative must vanish for some point u in (ab), and we have

$$fx_0 - fa = \frac{\zeta^{\frac{1}{2}}(a_0, \ldots, a_n, x_0)}{\zeta^{\frac{1}{2}}(a_0, \ldots, a_n)} \frac{f^n u}{(n+1)!} \cdot \frac{1}{u - \dfrac{1}{n+1} \dfrac{\begin{vmatrix} 1 & a_0, \ldots, a_0^{n-1}, a_0^{n+1} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ 1 & a_n, \ldots, a_n^{n-1}, a_n^{n+1} \end{vmatrix}}{\zeta^{\frac{1}{2}}(a_0, \ldots, a_n)}}$$

$$= \frac{(x_0 - a_0) \ldots (x_0 - a_n)}{(n+1)!} f^n u \frac{1}{u - \Delta^{n|h} a^{n+1}/(n+1)!\, h^n} .$$

In this, we have

$$\Delta^{n|h} a^{n+1}/(n+1)!\, h^n = \tfrac{1}{2}(a + b) .$$

$$\mathop{\pounds}_{n=\infty} \frac{(x_0 - a_0) \ldots (x_0 - a_n)}{(n+1)!} = 0 .$$

Also, we have $f^n u = 0$ by the preceding corollary. But, u being an unknown point in (ab), and $h = (b - a)/n$, we do not know but that for $n = \infty$, we may have (as we probably do),

$$u = \Delta^{n|h} a^{n+1}/(n+1)!\, h^n = \tfrac{1}{2}(a + b) ,$$

and therefore cannot say that $fx_0 = fa$ when $n = \infty$. In point of fact, if we let n be constant and let h converge to zero u converges to a, and so also does

$$\mathop{\pounds}_{h=0} \frac{\Delta^{n|h} a^{n+1}}{(n+1)!\, h^n} = \left[\frac{d}{dx}\right]^n_{x=a} \frac{x^{n+1}}{(n+1)!} = a .$$

III.

Monomorphic Functions. Infinite Series.

§ 10. A holomorphic function fx is said to be monomorphic throughout an interval $(\alpha\beta)$, when for all points x, a, and u in that interval, we have

$$\mathop{\pounds}_{n=\infty} \frac{(x-a)^n}{n!} f^n u = 0 . \qquad (u \text{ between } x \text{ and } a)$$

This interval of monomorphism will be called the *region* of the function. The following corollaries flow from the preceding theorems :—

A monomorphic function cannot have an infinite number of equal values in any finite interval (however small) without being constant throughout its entire region. It cannot have an infinite number of equal values in any infinitesimal interval, that is it cannot have a multiple point of infinite multiplicity, without being constant throughout its entire region.

In particular, it cannot have an infinite number of zeros in any finite or infinitesimal (zero of infinite nullitude) interval without vanishing throughout its region.

If a monomorphic function has an infinity of equal values, zeros, a point of infinite multiplicity, or a zero of infinite nullitude; then all the successive derivatives of the function vanish throughout its region.

Two monomorphic functions fx and φx which are equal over any finite interval, are equal all over their common region of monomorphism. For their difference vanishes an infinite number of times in the equality interval.

THEOREM XI. *If two holomorphic functions φx and ψx have a common zero a, then the ratios $\varphi x/\psi x$ and $\varphi' x/\psi' x$ converge to the same limit as x converges to a.*

Let $x = a + h$, then

$$\frac{\varphi x}{\psi x} = \frac{\Delta\varphi a}{\Delta\psi a} = \frac{\Delta\varphi a/h}{\Delta\psi a/h}.$$

Hence

$$\pounds_{x=a} \frac{\varphi x}{\psi x} = \pounds_{h=0} \frac{\Delta\varphi a/h}{\Delta\psi a/h} = \frac{\varphi' a}{\psi' a}$$

$$= \pounds_{x=a} \frac{\varphi' x}{\psi' x}.$$

If a is a common zero of $\varphi' x$ and $\psi' x$, then the ratio $\varphi x/\psi x$ converges to the same limit as does $\varphi'' x/\psi'' x$ when x converges to a, as is easily shown by a repetition of the above. Generally, if the first r derivatives of φx and ψx have the common zero a with φx and ψx, then the ratios $\varphi x/\psi x$ and $\varphi^{r+1}x/\psi^{r+1}x$ converge to the same limit as x converges to a.

§ 11. THEOREM XII. *When a function is monomorphic throughout a certain finite interval $(\alpha\beta)$ containing the point a, it can be expanded in an infinite series of positive integral powers of $x - a$, converging for all points within $(\alpha\beta)$.*

First Proof:—Let b be a fixed point in $(\alpha\beta)$. Divide the interval (ab) into n equal parts equal h. For convenience let

$$E^r fa = f(a + rh). \qquad (r = 0, \ldots, n)$$

The function

$$Fx \equiv \begin{vmatrix} fx & 1 & x & \ldots & x^n \\ E^0 fa & 1 & E^0 a & \ldots & E^0 a^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ E^n fa & 1 & E^n a & \ldots & E^n a^n \end{vmatrix} \div \begin{vmatrix} 1 & E^0 a & \ldots & E^0 a^n \\ \cdot & \cdot & \cdot & \cdot \\ 1 & E^n a & \ldots & E^n a^n \end{vmatrix}$$

is a holomorphic (n finite) in the interval $(\alpha\beta)$, which has the $n+1$ zeros

$$a + rh . \qquad (r = 0, \ldots, n)$$

Consequently

$$Fx = (x-a)(x-a-h)\ldots(x-b)\frac{F^{n+1}u}{(n+1)!}$$

$$= (x-a)(x-a-h)\ldots(x-b)\frac{f^{n+1}u}{(n+1)!},$$

u in (a, b, x). Let x be a point in the interval (ab). Then

$$\operatorname{mod} Fx < \frac{(a-b)^{n+1}}{(n+1)!} f^{n+1}u ,$$

which vanishes when $n = \infty$ since fx is monomorphic. Fx vanishing throughout (ab) $(n=\infty)$, vanishes throughout $(\alpha\beta)$.

In the determinant form of Fx, regard the terms of each column as forming a sequence. Begin with the second term from the top and *completely difference* each column. This will be called the *complete difference* of the determinant. The complete differencing of any determinant does not alter its value. After forming the complete difference of the determinants in Fx, divide the numerator and denominator by

$$h^{\frac{1}{2}n(n+1)}/n\,!! ,$$

distributed as shown below. We obtain

$$Fx \equiv \begin{vmatrix} fx & 1 & \frac{x}{1!} & \ldots & \frac{x^n}{n!} \\ fa & 1 & \frac{a}{1!} & \ldots & \frac{a^n}{n!} \\ \frac{\Delta' fa}{h} & 0 & \frac{\Delta' a}{1!\,h} & \ldots & \frac{\Delta' a^n}{n!\,h} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\Delta^n fa}{h^n} & 0 & \frac{\Delta^n a}{1!\,h^n} & \ldots & \frac{\Delta^n a^n}{n!\,h^n} \end{vmatrix} \div \begin{vmatrix} \frac{\Delta' a}{1!\,h} & \ldots & \frac{\Delta' a^n}{n!\,h^n} \\ \cdot & \cdot & \cdot \\ \frac{\Delta^n a}{1!\,h} & \ldots & \frac{\Delta^n a^n}{n!\,h^n} \end{vmatrix} .$$

As n becomes infinitely large $h = (a - b)/n$ becomes infinitely small. Hence throughout $(\alpha\beta)$, we have

$$\begin{vmatrix} fx & 1 & x & x^2/2! & \dots \\ fa & 1 & a & a^2/2! & \dots \\ f'a & 0 & 1 & a/1! & \dots \\ f''a & 0 & 0 & 1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0,$$

or

$$fx = fa + (x-a)f'a + \frac{(x-a)^2}{2!}f''a + \dots$$

Lemma. The expansion of the above determinant is effected by means of the identity

$$A_n \equiv \begin{vmatrix} 1 & \frac{x}{1!} & \dots & \frac{x^n}{n!} \\ 1 & \frac{a}{1!} & \dots & \frac{a^n}{n!} \\ 0 & 1 & \dots & \frac{a^{n-1}}{(n-1)!} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots\ 1, & a \end{vmatrix} \equiv \frac{(a-x)^n}{n!}.$$

For,

$$A_n \equiv C_n - C_{n-1}x + C_{n-2}\frac{x^2}{2!} - \dots + (-1)^n \frac{x^n}{n!},$$

wherein

$$C_r \equiv \begin{vmatrix} \frac{a}{1!} & \dots & \frac{a^r}{r!} \\ 1 & \dots & \frac{a^{r-1}}{(r-1)!} \\ \cdot & \cdot & \cdot \\ 0 & \dots\ 1, & a \end{vmatrix}.$$

$C_r = a^r/r!$, for $r = 1, 2, 3$. Suppose this is true for $p - 1$. Then

$$\begin{aligned} C_p &= C_{p-1}a - C_{p-2}\frac{a^2}{2!} + \dots + (-1)^{p+1}\frac{a^r}{r!} \\ &= \frac{a^{p-1}}{(p-1)!}\frac{a}{1!} - \frac{a^{p-2}}{(p-2)!}\frac{a^2}{2!} + \dots + (-1)^{p+1}\frac{a^0}{0!}\frac{a^p}{p!} \\ &= \frac{a^p}{p!} - \left[\frac{a^p}{p!}\frac{a^0}{0!} - \frac{a^{p-1}}{(p-1)!}\frac{a^1}{1!} + \dots + (-1)^p\frac{a^0}{0!}\frac{a^p}{p!}\right]. \end{aligned}$$

The second term of the second member is zero, being

$$\frac{a^p}{p!}(1-1)^p.$$

Hence, $C_r = a^r/r!$ for all integral values of r, and we have

$$A_n = \frac{a^n}{n!} - \frac{a^{n-1}}{(n-1)!}\frac{x^1}{1!} + \ldots + (-1)^n \frac{x^n}{n!},$$

$$= \frac{(a-x)^n}{n!}.$$

Otherwise and more simply (the use of the lemma may be avoided) thus; multiply the $(r+2)$th row by $(x-a)^r/r!$ $(r = 1, 2, 3, \ldots)$, and divide the corresponding columns by the same quantities. Subtract each row below the first from the first. All elements in the first row vanish except the first, which is

$$fx - fa - \Sigma(x-a)^r f^r a/r!.$$

All terms of the determinant vanish except the diagonal term, and each element of the diagonal is unity except the first as written above.

Second Proof: The monomorphic function Fx has the $n+1$ zeros $a + rh$ $(r = 0, \ldots, n)$. Therefore by Theorem VIII, we have

$$Fx = (x-a)\ldots(x-a-nh)\frac{F^{n+1}u}{(n+1)!}.$$

Let n remain constant and let h converge to zero. Whence

$$\begin{vmatrix} fx & 1 & \frac{x}{1!} & \ldots & \frac{x^n}{n!} \\ fa & 1 & \frac{a}{1!} & \ldots & \frac{a^n}{n!} \\ f'a & 0 & 1 & \ldots & \frac{a^{n-1}}{(n-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a & 0 & 0 & \ldots & 1 \end{vmatrix} = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}u,$$

or

$$fx = fa + \frac{(x-a)^1}{1!} f'a + \ldots + \frac{(x-a)^n}{n!} f^n a + \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}u.$$

Since fx is monomorphic the last term vanishes when $n = \infty$, and the infinite series is equal to fx throughout $(\alpha\beta)$.

Third Proof:—Let $a_1, \ldots, a_n$, and a be *any* points in $(\alpha\beta)$. The function

$$Fx \equiv \begin{vmatrix} fx & 1 & \frac{x}{1!} & \cdots & \frac{x^n}{n!} \\ fa & 1 & \frac{a}{1!} & \cdots & \frac{a^n}{n!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n & 1 & \frac{a_n}{1!} & \cdots & \frac{a_n^n}{n!} \end{vmatrix} \div \begin{vmatrix} 1 & \frac{a}{1!} & \cdots & \frac{a^n}{n!} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{a_n}{1!} & \cdots & \frac{a_n^n}{n!} \end{vmatrix},$$

has the $n+1$ zeros $a, a_1, \ldots, a_n$. Therefore by Theorem VIII we have

$$Fx = (x-a)\ldots(x-a_n)\frac{F^{n+1}u}{(n+1)!}.$$

But the value of Fx given by the above ratio is a function of $a_1, \ldots, a_n$, and takes the indeterminate form $0/0$ when $a_1, \ldots, a_n$ converge to a as a limit. To evaluate this limit we apply the method of Theorem XI by operating on the numerator and denominator of the ratio with

$$\left[\frac{\partial}{\partial a_1}\right]^1_{a_1=a} \cdots \left[\frac{\partial}{\partial a_n}\right]^n_{a_n=a},$$

which produces identically the same result as the last proof.

Fourth Proof: (After Homersham Cox and Cauchy).

The holomorphic function

$$Fx \quad \begin{vmatrix} fx & 1 & \frac{x}{1!} & \cdots & \frac{x^n}{n!} \\ fa & 1 & \frac{a}{1!} & \cdots & \frac{a^n}{n!} \\ f'a & 0 & 1 & \cdots & \frac{a^{n-1}}{(n-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^na & 0 & 0 & \cdots & 1 \end{vmatrix},$$

and its first n derivatives vanish at a. Change x into x_0, where x_0 is any arbitrary fixed point (not a) in $(\alpha\beta)$. Consider the function

$$Jx \quad (x_0-a)^{n+1}Fx - (x-a)^{n+1}Fx_0.$$

This function is holomorphic in $(\alpha\beta)$ and has the zeros a and x_0. Its first n derivatives have the common zero a. Therefore by Theorem VII its

$(n+1)$th derivative must have a zero u, between a and x_0. Hence

$$(x_0 - a)^{n+1} F^{n+1}u - (n+1)!\, Fx_0 = 0\,,$$

or

$$Fx = \frac{(x-a)^{n+1}}{(n+1)!} F^{n+1}u$$

$$= \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}u\,.$$

Since x_0 is any point in $(a\beta)$.

*Fifth Proof:** We may avoid the expansion of the determinant given in the lemma of the first proof, as follows:—

The function

$$Fx \equiv \begin{vmatrix} fx & 1 & x & \dots & x^n \\ fa & 1 & a & \dots & a^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n & 1 & a_r & \dots & a_n^n \end{vmatrix}$$

$$\equiv \zeta fx + \varphi x\,,$$

wherein $\zeta = \zeta^{\frac{1}{2}}(a, a_1, \dots, a_n)$ and φx is a rational integral function of degree n, has the zeros $a, a_1, \dots, a_n$ in $(a\beta)$, and must have the same interval of monomorphism with fx. We have, x and $x+h$ in $(a\beta)$,

$$F(x+h) = \zeta f(x+h) + \varphi(x+h)$$

$$= \zeta f(x+h) + \varphi x + \frac{h}{1!}\varphi' x + \dots + \frac{h^n}{n!}\varphi^n x\,. \qquad \text{(i)}$$

Since Fx has $n+1$ zeros its nth derivative has a zero among them. Therefore we have

$$F^r x = \zeta f^r x + \varphi^r x\,, \qquad (r = 0, \dots, n-1)$$

and

$$F^n u = \zeta f^n u + \varphi^n u = 0\,.$$

Multiply these $n+1$ equations by $h^r/r!$ $(r = 0, \dots, n)$ and add them together. Subtract the resulting equation from (i), whence results

$$f(x+h) - \sum_{r=0}^{n-1} \frac{h^r}{r!} f^r x - \frac{h^n}{n!} f^n u = \frac{1}{\zeta}\left[F(x+h) - \sum_{r=0}^{n-1}\frac{h^r}{r!}F^r x\right].$$

---

* American Journal of Mathematics, July, 1893.

Let $a = x$ and $a_n = x + h$. Then Fx and $F(x + h)$ vanish, and the second member of this equation becomes

$$-\frac{\sum\limits_{r=1}^{n-1} \frac{h^r}{r!} F^r x}{\zeta^{\frac{1}{2}}(x, a_1, \ldots, a_{n-1}, x + h)},$$

which takes the form $0/0$ as $a_1, \ldots, a_{n-1}$ converge to x. To evaluate this limit, apply to each term of the ratio, the operator

$$\left[\frac{\partial}{\partial a_1}\right]^1_{a_1 = x} \cdots \left[\frac{\partial}{\partial a_{n-1}}\right]^{n-1}_{a_{n-1} = x}.$$

Since $\left[\frac{\partial}{\partial a_r}\right]^r_{a_r = x}$ causes $F^r x$ to vanish ($r = 1, \ldots, n - 1$), the numerator of this ratio is zero in the limit. While

$$\left[\frac{\partial}{\partial a_1}\right]^1_{a_1 = x} \cdots \left[\frac{\partial}{\partial a_{n-1}}\right]^{n-1}_{a_{n-1} = x} \zeta^{\frac{1}{2}}(x, a_1, \ldots, a_{n-1}, x + h) = (n - 1)!!\, h^n.$$

Consequently, the limit of the ratio in question is zero, and we have

$$f(x + h) = fx + hf'x + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}x + \frac{h^n}{n!} f^n u,$$

which is unconditionally convergent and equal to fx when $n = \infty$ for all values of x and $x + h$ in $(\alpha\beta)$.

COROLLARY. If zero is a point of the interval $(\alpha\beta)$, then putting $a = 0$

$$fx = f0 + xf'0 + \ldots + \frac{x^n}{n!} f^n 0 + \frac{x^{n+1}}{(n+1)!} f^{n+1} u,$$

as a particular case, and as before we may make $n = \infty$.

Also, when zero is a point in $(\alpha\beta)$, we have

$$f0 = fa - af'a + \frac{a^2}{2!} f''a + \ldots,$$

by putting $x = 0$.

THEOREM XIII. *The rth derivative of the series*

$$fa + \frac{(x-a)^1}{1!} f'a + \frac{(x-a)^2}{2!} f''a + \ldots \text{ ad. inf.}$$

formed by taking the sum of the rth derivatives of each term, is equal to the rth derivative of the function fx for all points in the interval $(\alpha\beta)$ of monomorphism of the function.

For, the rth derivative of the monomorphic function

$$Fx \equiv \begin{vmatrix} fx & 1 & x & \ldots & x^n/n! \\ fa & 1 & a & \ldots & a^n/n! \\ f'a & 0 & 1 & \ldots & a^{n-1}/(n-1)! \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^na & 0 & 0 & \ldots & 1 \end{vmatrix}$$

is holomorphic in the interval $(\alpha\beta)$. This derivative F^rx $(r < n)$ has the zero a, and its first $n - r$ derivatives also have the zero a. Let x_0 be a fixed value in $(\alpha\beta)$. Consider the function

$$Jx \equiv \frac{(x_0 - a)^{n+1}}{(n+1)!} F^rx - \frac{(x-a)^{n+1}}{(n+1)!} F^rx_0$$

This function is monomorphic throughout the interval $(\alpha\beta)$. It has the zeros a and x_0 and its first $n - r$ derivatives have the common zero a. Consequently, its $(n + 1 - r)$th derivative must have a zero (u) between x_0 and a, and we have

$$\frac{(x_0 - a)^{n+1}}{(n+1)!} F^{n+1}u - \frac{(u-a)^r}{r!} F^rx_0 = 0 ,$$

or since x_0 is any point in $(\alpha\beta)$

$$\begin{aligned} F^rx &= \frac{r!}{(u-a)^r} \frac{(x-a)^{n+1}}{(n+1)!} F^{n+1}u \\ &= \frac{r!}{(u-a)^r} \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}u , \\ &= \frac{r!}{\theta^r} \frac{(x-a)^{n+1-r}}{(n+1)!} f^{n+1}u , \qquad (0 < \theta < 1) \end{aligned}$$

which vanishes when $n = \infty$ for any finite value r. Therefore

$$f^rx = f^ra + \frac{(x-a)^1}{1!} f^{r+1}a + \frac{(x-a)^2}{2!} f^{r+2}a + \ldots \text{ ad. inf.}$$

for all values of x in $(\alpha\beta)$.

We observe, that we may differentiate the series with remainder after nth term as given in the second proof of XII, just as though u remained constant during the operation. For, we may write

$$Jx \equiv \frac{(x_0 - a)^{n-r+1}}{(n-r+1)!} F^rx - \frac{(x-a)^{n-r+1}}{(n-r+1)!} F^rx_0 .$$

From which we obtain

$$F^r x = \frac{(x-a)^{n+1-r}}{(n+1-r)\,!} f^{n+1} u\,,$$

which is a different form of the remainder from that obtained above, leading to the same result, and which is what would be obtained by differentiating

$$Fx = \frac{(x-a)^{n+1}}{(n+1)\,!} f^{n+1} u$$

r times. The u's in the two forms of course not having the same value.

THEOREM XIV. *If a function fx be monomorphic throughout any finite interval $(\alpha\beta)$, it can be expanded in an infinite series of positive integral powers of x, converging for all points in $(\alpha\beta)$.*

Expanding the determinantal form of the second proof of Theorem XI, by its first row, we have

$$fx = A_0 + A_1 x + A_2 \frac{x^2}{2!} + \ldots + A_n \frac{x^n}{n!} + \frac{(x-a)^{n+1}}{(n+1)\,!} f^{n+1} u\,,$$

wherein

$$A_r \equiv \begin{vmatrix} f^r a & \frac{a}{1\,!} & \ldots & \frac{a^{n-r}}{(n-r)\,!} \\ f^{r+1} a & 1 & \ldots & \frac{a^{n-r-1}}{(n-r-1)\,!} \\ \cdot & \cdot & \cdot & \cdot \\ f^n a & 0 & \ldots & 1 \end{vmatrix}$$

$$\equiv f^r a - a f^{r+1} a + \frac{a^2}{2!} f^{r+2} a + \ldots + (-1)^{n-r} \frac{a^{n-r}}{(n-r)\,!} f^n a\,.$$

The series is evidently convergent when $n = \infty$, for all values of x in $(\alpha\beta)$.

COROLLARY 1. If zero be a point in $(\alpha\beta)$, then putting $a = 0$ we reduce the series as before to that of Corollary of Theorem XII. (Maclaurin's series).

COROLLARY 2. In illustration, we notice that if zero be a point in $(\alpha\beta)$, then in virtue of Bernoulli's series derivable directly from XII, we have

$$A_r = f^r 0 = f^r a - a f^{r+1} a + \frac{a^2}{2!} f^{r+2} 0 - \ldots \text{ ad. inf.}$$

Therefore the above series may be written in the particular form of Maclaurin's series

$$fx = f0 + x f'\,0 + \ldots + \frac{x^n}{n!} f^n 0 + \ldots$$

whenever zero is a point in $(\alpha\beta)$ and not otherwise (Cor. XII).

§ 12. An even function is one which does not change its value when the sign of the argument is changed. Thus

$$fa = f(-a).$$

An odd function is one which changes its sign but not its absolute value, when the sign of the argument is changed. Thus

$$fa = -f(-a).$$

THEOREM XV. *An even (odd) monomorphic function can be expanded in an infinite series of positive even (odd) integral powers of the variable for all points in the interval of monomorphism.*

By the preceding theorem, we have

$$f(x) = A'_0 + A'_1 x + A'_2 x^2 + \dots,$$

$$f(-x) = A'_0 - A'_1 x + A'_2 x^2 - \dots.$$

By addition, we have, if the function is even

$$fx = A'_0 + A'_2 x^2 + A'_4 x^4 + \dots.$$

By subtraction, we have, if the function is odd

$$fx = A'_1 x + A'_3 x^3 + A'_5 x^5 + \dots.$$

COROLLARY 1. The derivatives of even order of an even (odd) function are even (odd) functions. The derivatives of odd order of an even (odd) function are odd (even) functions, and are expressible, when the functions are monomorphic, in infinite series of integral powers which are obtained by differentiating the infinite series of the function.

COROLLARY 2. If zero is a point in the monomorphic interval of fx, then at zero $fx = a_0$ if fx is even. $fx = 0$ if fx is odd.

COROLLARY 3. A periodic function is defined as one which repeats its values in the same order in successive equal intervals, thus

$$fx = f(x \pm rp)$$

for all integral values of r, p being the constant interval or period of the function. It follows from the above that

$$f^n x = f^n(x \pm rp)$$

if fx is holomorphic.

If fx is an even (odd) periodic monomorphic function, its derivatives of even (odd) order are even (odd) periodic monomorphic functions, and derivatives of odd (even) order are odd (even) functions.

The derivatives of odd (even) order of an even (odd) periodic holomorphic function vanish at the points $\pm\, rp$ $(r = 0, \ldots, n)$.

§ 13. THEOREM XVI. *On the expansion of a monomorphic function fx in terms of an infinite series of monomorphic functions $\varphi_r x$ $(r = 0, 1, 2, \ldots)$ whose law of formation with respect to r is given.*

Let fx and $\varphi_r x$ be monomorphic functions throughout an interval $(\alpha\beta)$. The object of the investigation is to effect the design of a convergent series

$$\sum_{r=0}^{\infty}{}' A_r \varphi_r x \,,$$

(in which the coefficients A_r are independent of x) such that the difference between the series and the function fx shall vanish throughout $(\alpha\beta)$.

Consider the function

$$\sum_{r=0}^{n}{}' A_r \varphi_r x \,,$$

in which, in order to provide an absolute term, we put $\varphi_0 x \equiv 1$.

Let

$$F(x, n) \equiv fx - \sum_{r=0}^{n}{}' A_r \varphi_r x \,.$$

Let $a, a_1, \ldots, a_n$ be points in $(\alpha\beta)$. Then at these points we have

$$F(a_r, n) = fa_r - \sum_{r=0}^{n}{}' A_r \varphi_r a_r \,. \qquad (r = 0, \ldots, n)$$

In these $n + 1$ equations we have the $n + 1$ undetermined values A_r $(r = 0, \ldots, n)$. Let the conditions which determine these be

$$F(a_r, n) = 0 \,. \qquad (r = 0, \ldots, n)$$

Then we have

$$F(x, n) \equiv \begin{vmatrix} fx & \varphi_0 x & \cdots & \varphi_n x \\ fa & \varphi_0 a & \cdots & \varphi_n a \\ \cdot & \cdot & \cdot & \cdot \\ fa_n & \varphi_0 a_n & \cdots & \varphi_n a_n \end{vmatrix} \div \begin{vmatrix} \varphi_0 a & \cdots & \varphi_n a \\ \cdot & \cdot & \cdot \\ \varphi_0 a_n & \cdots & \varphi_n a_n \end{vmatrix}, \qquad \text{(i)}$$

$$\equiv fx - \sum_{r=0}^{n}{}' A_r \varphi_r x \,,$$

in which A_r is independent of x and is determined by expanding the determinant with respect to its first row.

First:—We treat the function $F(x, n)$ as given by (i) in the following independent manner:* Let a and a_n be two points in $(\alpha\beta)$ and let

$$a_r = a + rh\,, \qquad (r = 0, \ldots, n)$$

so that $h = (a_n - a)/n$.

Completely difference the two determinants in (i), beginning with the second row in the numerator and the first in the denominator. Then divide the row of pth differences by $h^p\ (p = 1, \ldots, n)$, in the two terms of the ratio. These operations do not change the value of $F(x, n)$. Now let a_n converge to a as a limit, h converging to zero. We have

$$F(x, n)_{a_r = a} = \frac{\begin{vmatrix} fx & 1 & \varphi_1 x & \ldots & \varphi_n x \\ fa & 1 & \varphi_1 a & \ldots & \varphi_n a \\ f'a & 0 & \varphi_1' a & \ldots & \varphi_n' a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a & 0 & \varphi_1^n a & \ldots & \varphi_n^n a \end{vmatrix}}{|\varphi_1' a \ldots \varphi_n^n a|}\,. \qquad \text{(ii)}$$

---

* This method holds good for holomorphic functions of a complex variable as well as for monomorphic functions of a real variable. For, let f and φ_r be holomorphic functions of z throughout a certain area C. The functions are expansible in Taylor's series throughout C. Let R_f and R_{ϕ_r}, be the remainders after the nth term in Taylor's expansion of these functions. Then by the same method of proof as above we get ($\varphi_0 z$ 1),

$$\frac{\begin{vmatrix} fz & 1 & \varphi_1 z & \ldots & \varphi_n z \\ fa & 1 & \varphi_1 a & \ldots & \varphi_n a \\ f'a & 0 & \varphi'_1 a & \ldots & \varphi'_n a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a & 0 & \varphi_1^n a & \ldots & \varphi_n^n a \end{vmatrix}}{|\varphi'_1 a \ldots \varphi_n^n a|} = \frac{\begin{vmatrix} R_f & R_{\phi_1} & \ldots & R_{\phi_n} \\ f'a & \varphi_1' a & \ldots & \varphi'^n a \\ \cdot & \cdot & \cdot & \cdot \\ f^n a & \varphi_1^n a & \ldots & \varphi_n^n a \end{vmatrix}}{|\varphi_1' a \ldots \varphi_n^n a|}\,,$$

or

$$fz - fa - \sum_{r=1}^{n} A_r \varphi_r z = R_f - \sum_{r=1}^{n} A_r R_{\phi_r}\,.$$

If $\sum_{r=1}^{\infty} A_r$ be convergent then the second member vanishes when $n = \infty$, since R_f and R_{ϕ_r} vanish. Therefore

$$fz = fa + \sum_{r=1}^{\infty} A_r \varphi_r x$$

for all values of z in C. In like manner we show that

$$f^m z = \sum_{r=1}^{\infty} A_r \varphi_r^m z$$

throughout C.

In each term of this ratio multiply the row of pth derivatives by $(x-a)^p/p\,!$, and subtract each row in the numerator (below the first) from the first row. The value of $F(x, n)_{a_r=a}$ is unchanged, and the element of the first row which was $\varphi_r x$, is now

$$\varphi_r x - \varphi_r a - (x-a)\,\varphi'_r a - \ldots - \frac{(x-a)^n}{n\,!}\,\varphi_r^n a\,.$$

Since $\varphi_r x$ is monomorphic, this last expression is equal to

$$\frac{(x-a)^{n+1}}{(n+1)\,!}\,\varphi_r^{n+1} u_r\,,$$

u_r being between x and a. Thus the value of $F(x, n)_{a_n=a}$ is

$$\frac{\begin{vmatrix} \frac{(x-a)^{n+1}}{(n+1)\,!} f^{n+1}u, & \frac{(x-a)^{n+1}}{(n+1)\,!}\,\varphi_1^{n+1}u_1, & \ldots, & \frac{(x-a)^{n+1}}{(n+1)\,!}\,\varphi_n^{n+1}u \\ (x-a)\,f'a & , (x-a)\,\varphi_1'a & , \ldots, & (x-a)\,\varphi'_n a \\ \cdot & \cdot & \cdot & \cdot \\ \frac{(x-a)^n}{n\,!} f^n a & , \frac{(x-a)^n}{n\,!}\,\varphi_1^n a & , \ldots, & \frac{(x-a)^n}{n\,!}\,\varphi_n^n a \end{vmatrix}}{\begin{vmatrix} \frac{(x-a)'}{1\,!}\,\varphi_1'a, \ldots, \frac{(x-a)^n}{n\,!}\,\varphi_n^n a \end{vmatrix}}$$

$$= \frac{(x-a)^{n+1}}{(n+1)\,!} f^{n+1}u + \sum_{p=1}^{n}{}' B_p \frac{(x-a)^p}{p\,!} f^p a\,.$$

When $n=\infty$, the first term of this second member vanishes. Let us examine the second term. We have,

$$\sum_{p=1}^{\infty}{}' \frac{(x-a)^p}{p\,!} f^p a = fx - fa\,,$$

while

$$B_p = \frac{W_p}{W}\,,$$

wherein W_p is the determinant

$$W = \left.\frac{(x-a)'}{1\,!}\,\varphi_1'a, \ldots, \frac{(x-a)^n}{n\,!}\,\varphi_n^n a\,\right|\,,$$

with its pth row replaced by

$$\frac{(x-a)^{n+1}}{(n+1)\,!}\,\varphi_1^{n+1}u_1\,, \;\ldots,\; \frac{(x-a)^{n+1}}{(n+1)\,!}\,\varphi_n^{n+1}u_r\,.$$

Expanding both W and W_p with respect to their pth rows, we have

$$B_p = \frac{\sum_{r=1}^{n} (-1)^{r+1} \frac{W_{rp}}{W} \frac{(x-a)^{n+1}}{(n+1)!} \varphi_r^{n+1} u_r}{\sum_{r=1}^{n} (-1)^{r+1} \frac{W_{rp}}{W} \frac{(x-a)^p}{p!} \varphi_r^p a},$$

wherein W_{rp} is the rth pth minor of W. The ratio W_{rp}/W in the ratio B_p may be taken to be the ratio of the rth pth minor of the Wronskian

$$|\varphi_1'a, \ldots, \varphi_n^n a|$$

to the Wronskian. We will consider it as such and call this the *Wronskian-ratio* of the series. It is, as will be shown, the ratio upon which depends the convergency of the series and its equality with fx. For, if when $n = \infty$

$$\sum_{r=1}^{\infty} W_{rp}/W$$

is a convergent series, the numerator of B_p vanishes when $n = \infty$. This condition includes the condition that the Wronskian $|\varphi_1'a, \ldots, \varphi_n^n a|$ shall not vanish, or that there must not be a linear relation between the functions

$$\varphi_1'x, \ldots, \varphi'_n x$$

at the point a. This condition is sufficient but not necessary.

Under this condition we assert that the function $F(x, \infty)$ of (ii) vanishes throughout $(\alpha\beta)$ and we have

$$fx - fa = \sum_{r=1}^{\infty} A_r (\varphi_r x - \varphi_r a),$$

wherein

$$A_r = (-1)^{r+1} \sum_{p=1}^{\infty} (-1)^{p+1} \frac{W_{rp}}{W} f^p a.$$

Second:—The function $F(x, n)$ of (i) is holomorphic in the interval $(\alpha\beta)$ and has the zeros $a, a_1, \ldots, a_n$. Therefore

$$F(x, n) = (x-a)\ldots(x-a_n)\left[\frac{\partial}{\partial x}\right]_{x=u}^{n+1} \frac{F(x, n)}{(n+1)!}.$$

When the a's converge to a, this becomes

$$F(x, n)_{a_r = a} = \frac{(x-a)^{n+1}}{(n+1)!} F^{n+1}(u, n),$$

u between x and a. But the determinant ratio (i) takes the form $0/0$ when the

a's converge to a. To remove this indetermination we apply to the numerator and denominator, the operator

$$\left[\frac{\partial}{\partial a_1}\right]^1_{a_1=a} \cdots \left[\frac{\partial}{\partial a_n}\right]^n_{a_n=a}.$$

Whence results (ii) provided

$$|\varphi_1'a, \ldots, \varphi_n^n a|$$

does not vanish. Consequently

$$fx - fa - \sum_{r=1}^{n} A_r(\varphi_r x - \varphi_r a) = \frac{(x-a)^{n+1}}{(n+1)!}\left[f^{n+1}u - \sum_{r=1}^{n} A_r \varphi_r^{n+1} u\right], \quad \text{(iii)}$$

wherein

$$A_r = (-1)^{r+1} \sum_{p=1}^{n} (-1)^{p+1} \frac{W_{rp}}{W} f^p a .$$

$W = |\varphi_1'a \ldots \varphi_n^n a|$ and W_{rp} is the rth pth minor of W. u lies between x and a.

In general, in order that we may have

$$fx = fa + \sum_{r=1}^{\infty} A_r(\varphi_r x - \varphi_r a)$$

throughout $(\alpha\beta)$, it is sufficient that $\sum_{r=1}^{\infty} A_r$ shall be convergent. This condition depends on the convergency of the series $\sum_{p=1}^{\infty} W_{rp}/W$. We may now enunciate the theorem and say :—

The monomorphic function fx can always be expanded in an infinite series of monomorphic functions $\varphi_r x$ $(r = 0, 1, 2, \ldots)$, whenever

$$\sum_{r=1}^{\infty}\sum_{p=1}^{\infty} (-1)^{r+1}(-1)^{p+1}\frac{W_{rp}}{W} f^p a$$

is convergent. W_{rp} being the rth pth minor of the Wronskian

$$|\varphi_1'a, \ldots, \varphi_n^n a|.$$

The series being equal to fx for all points in their common region of monomorphism.

Corollary 1. When ΣA_r is convergent we have

$$fx = \sum_{r=0}^{\infty} A_r \varphi_r x$$

throughout the common region of the functions, also, the series has an unlimited number of derivatives which are equal to the corresponding derivatives of the function fx for all points throughout the region.

For, we have

$$F^m(x, n) = \frac{\begin{vmatrix} f^m x & \varphi_1{}^m x & \dots & \varphi_n^m x \\ f'a & \varphi_1'a & \dots & \varphi'_n a \\ \cdot & \cdot & \cdot & \cdot \\ f^n a & \varphi_1{}^n a & \dots & \varphi_n^n a \end{vmatrix}}{|\varphi_1'a, \dots, \varphi_n^n a|}$$

holomorphic in $(\alpha\beta)$. Let x_0 be an arbitrary fixed point in $(\alpha\beta)$. Consider the function

$$Jx = \frac{(x_0 - a)^{n+1-m}}{(n+1-m)!} F^m(x, n) - \frac{(x - a)^{n+1-m}}{(n+1-m)!} F^m(x_0, n).$$

Jx and its first $n - m$ derivatives vanish at a, Jx also vanishes at x_0. Therefore its $(n + 1 - m)$th derivative vanishes at some point u' between a and x_0. Consequently

$$F^m(x_0, n) = \frac{(x_0 - a)^{n+1-m}}{(n+1-m)!} F^{n+1-m}(u', n);$$

or, since x_0 is any point in $(\alpha\beta)$, we have

$$f^m x - \sum_{r=0}^{n} A_r \varphi_r^m x = \frac{(x-a)^{n+1-m}}{(n+1-m)!} \left[f^{n+1}u' - \sum_{r=1}^{n} A_r \varphi_r^{n+1} u' \right],$$

the second member of which vanishes when $n = \infty$ if $\sum^{\infty} A_r$ is convergent. We observe, from this result, that we may differentiate (iii) as though u were a constant.

Again, if we put

$$Jx = \frac{(x_0 - a)^{n+1}}{(n+1)!} F^m(x, n) - \frac{(x-a)^{n+1}}{(n+1)!} F^m(x_0, n),$$

we get

$$F^m(x, n) = \frac{m!}{(u' - a)^m} \frac{(x-a)^{n+1}}{(n+1)!} \left[f^{n+1}u' - \sum_{r=1}^{n} A_r \varphi_r^{n+1} u' \right]$$

which makes the vanishing of the second member with $n = \infty$ and ΣA_r convergent, more immediately evident.

Corollary 2. Let $R_f(x)$ be the value of the second member of (ii), and $R_\psi(x)$ be its value when we have the holomorphic function fx replaced by another holomorphic function ψx, having $(\alpha\beta)$ for a common region. Then will we have, in general,

$$R_f(x)/R_\psi(x) = \left[\frac{\partial}{\partial x}\right]_{x=u}^{n+1} R_f(x) \Big/ \left[\frac{\partial}{\partial x}\right]_{x=u}^{n+1} R_\psi(x)$$

wherein u lies between x and a, and $R_\psi x$ and $R_\psi^{n+1}(u)$ are not zero. Let

$$Jx \equiv R_\psi(x_0)\, R_f(x) - R_\psi(x)\, R_f(x_0)\,,$$

x_0 being any fixed point in $(\alpha\beta)$.

Jx is holomorphic in $(\alpha\beta)$ and vanishes together with its first n derivatives at a; it also vanishes at x_0. Its $(n+1)$th derivative must vanish at some point u between x_0 and a;

$$\therefore \qquad R_\psi(x_0)\, R_f^{n+1}(u) = R_f(x_0)\, R_\psi^{n+1}(u)\,.$$

Since x_0 is any point in $(\alpha\beta)$ we may write

$$R_f(x) = \frac{R_\psi(x)}{R_\psi^{n+1}(u)}\, R_f^{n+1}(u)\,.$$

We may if we choose let $R_\psi(x) \equiv \varphi_{n+1}x$. The particular case

$$R_\psi(x) = (x-a)^{n+1}/(n+1)!\,,$$

is

$$R_f(x) = \frac{(x-a)^{n+1}}{(n+1)!}\, R_f^{n+1}(u)\,.$$

Which is Theorem XVI.

COROLLARY 3. If fx is an even (odd) function, $\varphi_r x$ are even (odd) functions and conversely. That is to say fx has a common region with the even (odd) functions $\varphi_r x$ only when it is even (odd).

COROLLARY 4. If fx is an even (odd) function having a common interval, containing zero, with even (odd) functions $\varphi_r x$, then must the odd (even) derivative rows in the numerator and denominator of (ii) be omitted, when $a = 0$.

For these odd (even) derivatives all vanish at zero. Therefore, if we leave them out of the determinants, we have as before, $F(x, n)_{a_n = a}$ vanishing as well as its first n derivatives at $x = 0$. Applying the method of the second proof, this function also vanishes at x_0, and we deduce the same results as before with the rows in question left out.

COROLLARY 5. In general, if any two rows have all the elements in one row equal for $x = a$, and these elements differ from the corresponding elements in the other row by an addition or factor constant, then one of them must be omitted. The proof is the same as above. If any number of such rows are so related, then all but one must be omitted. These last two corollaries do not present exceptions to the general theorem, but merely particularizations. The omission of these rows is merely a method of eliminating the indetermination caused by their presence.

COROLLARY 6. If $\varphi_r x$ $(r = 1, 2, 3, \ldots)$ be rational integral functions of degree r, then we have for the value of (ii)

$$F(x, n)_{a_n = a} = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1} u ,$$

and if fx is monomorphic in $(\alpha\beta)$, the series

$$fx = fa + \sum_{r=1}^{n} A_r (\varphi_r x - \varphi_r a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1} u$$

is unconditionally convergent in $(\alpha\beta)$, when $n = \infty$.

IV. METAMORPHIC FUNCTIONS.

§ 14. Let us call those functions *metamorphic*, which are holomorphic for finite values of the argument, but which cease to be holomorphic when the argument becomes infinite.

Thus, let $\varphi(a_r x)$ be a periodic function in which a_r is a function of the number r, such that a_r increases without limit along with r, but is finite for finite values of r. The function $\varphi(a_r x)$ is supposed holomorphic when r is finite, but will, in general, become indeterminate when $r = \infty$. Its rth derivative being generally an infinity of the rth infinitude when $r = \infty$. Thus

$$\left[\frac{\partial}{\partial x}\right]^r \varphi(a^r x) = a_r^r \varphi^r(a_r x) .$$

While the complexity of the remainder will generally render the quantitative study of the expansion of a holomorphic function in terms of metamorphic functions by the preceding method, difficult if not impossible, it is interesting and important that we should attack the problem, since the qualitative analysis is complete in itself and the results serve to illustrate the limitations which surround this general method in its applications to close analysis. Let us consider the following :

THEOREM XVII. *On the expansion of a holomorphic function in terms of an infinite series of metamorphic functions.*

Let $f(x+h)$ be a function which is holomorphic in a certain interval $(\alpha\beta)$ containing $x + h$ and h, and

$$\varphi_r(k + a_r b x) \qquad (r = 1, \ldots, n)$$

the series of metamorphic functions.

By the corollary to XVI, since the functions are all holomorphic for n finite, we have

$$R_f(x) = \frac{R_\psi(x)}{R_\psi^{n+1}(u)} R_f^{n+1}(u) ,$$

in which u lies between x and a.

Putting $a = 0$, we have

$$R_f(x) = f(x+h) - fh - \sum_{r=1}^{n} (-1)^{r+1} A_r \, [\varphi(k + a_r bx) - \varphi k]/a_r ,$$

$$R_f^{n+1}(u) = f^{n+1}u - \sum_{r=1}^{n} (-1)^{r+1} A_r b^{n+1} a_r^{\,n} \, \varphi^{n+1}(k + a_r bu) .$$

Letting $\psi x = \varphi(k + a_{n+1}bx)$, we have

$$R_\psi(x) = \varphi(k + a_{n+1}bx) - \varphi k - \sum_{=1}^{n} (-1)^{r+1} B_r \, [\varphi(k + a_r bx) - \varphi k]/a_r ,$$

$$R_\psi^{n+1}(u) = b^{n+1} a_{n+1}^{n+1} \varphi^{n+1}(k + a_{n+1}bu) - \sum_{r=1}^{n} (-1)^{r+1} B_r b^{n+1} a_r^{\,n} \varphi^{n+1}(k + a_r bu) .$$

$$A_r = \sum_{p=1}^{n} \frac{(-1)^{p+1}}{b^p} \frac{W_{rp}}{W} \frac{f^p h}{\varphi^p k} ,$$

$$B_r = \sum_{p=1}^{n} \frac{(-1)^{p+1}}{b^p} \frac{W_{rp}}{W} ,$$

$$W = \zeta^{\frac{1}{2}}(a_1, \ldots, a_n) ,$$

$$\frac{W_{rp}}{W} = (-1)^{r-1} P_{p+1} \frac{\Pi a_m}{\Pi(a_m - a_r)} . \quad (m = 1, 2, \ldots \infty) \operatorname{Ex} r$$

P_m is the sum of the products m at a time without repetition of the quantities a_m^{-1} $(m = 1, 2, \ldots, \infty)$ Ex r. $P_0 = 1$.

W_{rp}/W has a finite limit provided $\sum^{\infty} a_r^{-1}$ $(r = 1, 2, \ldots, \infty)$ is convergent. The condition for the series to infinity to be equal to $f(x+h)$ throughout $(\alpha\beta)$ is that

$$\mathop{\pounds}_{n=\infty} R_\psi(x) \frac{R_f^{n+1}(u)}{R_\psi^{n+1}(u)} = 0 ,$$

for all values of x and u in the interval under consideration.

The general condition is, of course, that

$$\sum_{r=1}^{\infty} (-1)^{r+1} A_r \, [\varphi(k + a_r bx) - \varphi \mathrm{k}]$$

must be convergent.

In particular, suppose

$$\sum_{r=1}^{\infty} (-1)^{r+1} B_r \, [\varphi(k + a_r bx) - \varphi k]/a_r ,$$

is not infinite. Let this be X_∞.

The remainder is the product of $\varphi(k + a_{n+1}bx) - X_n$ into

$$\frac{\dfrac{f^{n+1}u}{b^{n+1}a_{n+1}^{n+1}} - \dfrac{1}{a_{n+1}} \sum_{r=1}^{n} (-1)^{r+1} \left[\dfrac{a_r}{a_{n+1}}\right]^n A_r \varphi^{n+1}(k + a_r bu)}{\varphi^{n+1}(k + a_{n+1}bu) - \dfrac{1}{a_{n+1}} \sum_{r=1}^{n} (-1)^{r+1} \left[\dfrac{a_r}{a_{n+1}}\right]^n B_r \varphi^{n+1}(k + a_r bu)} .$$

We have

$$\mathcal{L}_{n=\infty} \left[\frac{a_r}{a_{n+1}}\right]^n = 0 ,$$

for finite values of r, and at most equal to unity for $r = n = \infty$. The convergence of the series in general must require

$$\mathcal{L}_{n=\infty} A_n = 0 .$$

If $\varphi(k + a_{n+1}bx)$ is not infinite and $\varphi^{n+1}(k + a_{n+1}bu)$ not zero when $n = \infty$, we may say that the remainder vanishes for $n = \infty$.

It does not appear that we can use the general form of the remainder to place close conditions on the general theorem, owing to the complexity of form due to its general character. It seems best to regard it as a qualitative basis from which we may investigate the particular case of functions fx and $\varphi_r x$ in given specific form. General observations are suggested by the forms in which the functions enter the general formulæ.

1. If φ_r functions are even (odd), f is to be even (odd).

2. If φ_r functions are even or odd periodic functions and k a point at which the odd or even derivatives vanish, then must the corresponding rows be omitted in the fundamental determinant, throwing out the corresponding terms of the series. In this case while the parameter a_r remains unchanged under the φ function signs, the Wronskian becomes

$$W = \zeta^{\frac{1}{2}} (a_1^2 \ldots a_n^2) ,$$

which renders possible the finite limit to W_{rp}/W ($n = \infty$), when $a_r = r$, since this limit in this case is dependent on the convergence of Σa_r^{-2}. This secures such expansions in which the φ functions have the forms, sin rx, cos rx, e^{irx}, etc.

In general the period points of the series will be points of discontinuity or non-holomorphism of the series and any three such consecutive points will generally determine the $(\alpha\beta)$ interval, and sometimes two of them fix this interval.

3. The coefficients A_r involve the successive derivatives of f at a specific point. If therefore the derivatives of f after the mth vanish, we should ex-

pect to find, as we actually do, only an mth contact between the function fx and the series $\Sigma A_r \varphi_r$ throughout the interval $(\alpha\beta)$ of equality. That is to say the m derivatives of the series equal the m derivatives of f, but derivatives after the mth of the series lose their dependence upon f and become indeterminate or infinite.

V. Expansion in Negative Powers.

§ 15. In closing this note, it may be interesting to apply this method in seeking an answer to the question: What must be the form of the expansion of a function fx in terms of negative powers of the variable, if such be possible?

Regarding the foregoing investigation to result in the general condition, in order that a function fx shall be expansible in an infinite series of functions $\varphi_r x$, we must have

$$\begin{vmatrix} fx & 1 & \varphi_1 x & \varphi_2 x & \cdots \\ fa & 1 & \varphi_1 a & \varphi_2 a & \cdots \\ f'a & 0 & \varphi_1' a & \varphi_2' a & \cdots \\ f''a & 0 & \varphi_1'' a & \varphi_2'' a & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Let us apply this in answer to the above question.

For the expansion of fx in terms of the functions $(x-a)^{-r}$ to exist, we must have

$$\begin{vmatrix} fx & 1 & (x-a)^{-1}, & (x-a)^{-2} & \cdots \\ fc & 1 & (c-a)^{-1}, & (c-a)^{-2} & \cdots \\ f'c & 0 & -(c-a)^{-2}, & 2(c-a)^{-3} & \cdots \\ f''c & 0 & +2(c-a)^{-3}, & +2.3(c-a)^{-4} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Factoring, we obtain

$$\begin{vmatrix} fx & 1 & \left[\dfrac{c-a}{x-a}\right], & \left[\dfrac{c-a}{x-a}\right]^2, & \cdots \\ fc & 1 & 1 & 1 & \cdots \\ -(c-a)f'c & 0 & 1 & 2 & \cdots \\ +(c-a)^2 f''c & 0 & 1.2 & 2.3 & \cdots \\ -(c-a)^3 f'''c & 0 & 1.2.3 & 2.3.4 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Completely difference this determinant, as follows: Begin with the second column and subtract each column from the succeeding one. In the resulting determinant, begin with the third column and repeat the operation, and so on, until all elements above the main diagonal vanish except those in the first row, which now are

$$fx, \quad 1, \quad \frac{c-x}{x-a}, \quad \left[\frac{c-x}{x-a}\right]^2, \quad \ldots$$

In this determinant begin with the fourth row and divide the rows by

$$1!\,2!, \quad 2!\,3!, \quad 3!\,4!, \quad \ldots$$

respectively; then begin with the fifth column and multiply the columns respectively by

$$2!, \quad 3!, \quad 4!, \quad \ldots$$

Whence results

$$\begin{vmatrix} fx & 1 & \frac{c-x}{x-a}, & 1!\left[\frac{c-x}{x-a}\right]^2, & 2!\left[\frac{c-x}{x-a}\right]^3, & \ldots \\ fc & 1 & 0 & 0 & 0 & \ldots \\ -(c-a)f'c & 0 & 1 & 0 & 0 & \ldots \\ +\frac{(c-a)^2}{1!\,2!}f''c & 0 & 1 & 1 & 0 & \ldots \\ -\frac{(c-a)^3}{2!\,3!}f'''c & 0 & \frac{1}{2!} & 1 & 1 & \ldots \\ +\frac{(c-a)^4}{3!\,4!}f^{\text{iv}}c & 0 & \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Expanding this with respect to the first row,* we obtain

$$fx = \sum_{r=0}^{\infty} \frac{1}{r!}\left[\frac{x-c}{x-a}\right]^r \left[\frac{d}{dx}\right]^{r-1}_{x=c} [(x-a)^r f'x].$$

The coefficient of $(x-c)^p/(x-a)^p$ in the expansion of the determinant is

$$\sum_{j=1}^{p} C_{p-1,j-1}\frac{(c-a)^j}{j!}f^j c \equiv \left[\frac{d}{dx}\right]^{p-1}_{x=c}\left[\frac{(x-a)^p}{p!}f'x\right].$$

Let

$$F_r x = \frac{(x-a)^r}{r!}f'x.$$

Then

$$F_{r+1}\,x = \frac{x-a}{r+1}F_r x.$$

* We thus alight in a curious manner on a particular case of Burmann's series, which was to be expected.

Differentiating r times,

$$(r+1)\,F_{r+1}^{r}x = rF_r^{r-1}x + (x-a)\,F_r^{r}x\,.$$

The ratio of convergence of the series is

$$\frac{x-c}{x-a}\,\frac{1}{1+1/r}\left[1+\frac{(c-a)^r\,F_r^{r}c/r\,!}{(c-a)^{r-1}\,F_r^{r-1}c/(r-1)\,!}\right].$$

We started out to obtain the form of the expansion of fx in terms of negative powers of $(x-a)$, but were led through the simplifying of the determinant to the expansion form of powers of $(x-c)/(x-a)$. This may be reconverted through the identity

$$\frac{x-c}{x-a}=1-\frac{c-a}{x-a}\,.$$

Whence by substitution and expansion, we have

$$\begin{aligned} fx &= \sum_{r=0}^{\infty}(-1)^r\left[\frac{c-a}{x-a}\right]^r\sum_{p=r}^{\infty}C_{p,r}\left[\frac{d}{dx}\right]^{p-1}_{x=c}\left[\frac{(x-a)^p}{p\,!}f'x\right] \\ &= \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!}\left[\frac{c-a}{x-a}\right]^r\sum_{p=r}^{\infty}\left[\frac{d}{dx}\right]^{p-1}_{x=c}\left[\frac{(x-a)^p}{(p-r)\,!}f'x\right], \end{aligned}$$

when the expansion in negative powers is possible.* Otherwise n must be written instead of ∞, and a residual term R added. We may investigate the remainder as before, when the character of fx is known.

§ 16. The consideration of general methods in the application of the Differential Calculus in this direction is important and interesting. Important because it is the working method for reaching practical results, interesting because it tends to show somewhat the limitations which surround investigation in this direction. It is the last resort of the physicist who is working for utilitarian results when all other methods fail, and offers him a direct method of procedure. Such was the path followed by Fourier when he discovered the "Open Sesame" to the Theory of Heat.

* This should also be the expansion of a holomorphic function of the complex variable z, for points outside of a circle about c with radius mod $(c-a)$. But the coefficients involve derivatives at the point c, which would require fz to be holomorphic not only outside of the circle but also *at* the center c.

On the other hand if we consider fz to be holomorphic outside of a circle with a as a center and radius $R <$ mod $(c-a)$. Then this should be the expansion of fz for points z outside of the circle with center a and radius mod $(c-a)$, and on the same side of a straight line with a, which is drawn normal to the straight line joining c and a at its mid-point, c now being in the area of holomorphism of the function.

University of Virginia, *August*, 1893.